NOTE ON THE SELF-DUALITY OF THE UNRESTRICTED DISTRIBUTIVE LAW IN COMPLETE LATTICES

BY

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ABSTRACT

In this article, a complete lattice in which meets are completely distributive with respect to joins is defined in such a way that the indefinitely wide class of all sets is not involved. It is proved that the concept of such a lattice is self-dual.

The following theorem is well-known:

(1) In a lattice, joins are distributive with respect to meets if and only if meets are distributive with respect to joins.

In his paper [1], George N. Raney proved a theorem containing the following statement:

(2) In a complete lattice, joins are completely distributive with respect to meets if and only if meets are completely distributive with respect to joins.

The author of the present note wishes to give another proof of (2), using a method similar to that by which (1) is usually proved.

NOTATION. L and I are sets, and A_i is a subset of L for every element i of I. $(c_i | i \in I)$ is the function on I whose value at i is c_i for every element i of I, and $\{c_i | i \in I\}$ is the range of this function. If f is a function, R_f is its range. If F is a mapping of I onto a set of sets, $\prod F$ is the cartesian product of F, i.e., the set of all mappings f of I such that $f(i) \in F(i)$ for every element i of I. The sign $\prod_{i \in I} A_i$ means the same as the sign $\prod(A_i | i \in I)$. If X is a subset of a complete lattice, $\cap X$ and $\cup X$ are, respectively, the meet and the join of X. If $a_i (i \in I)$ are elements of a complete lattice, the signs $\cap_{i \in I} a_i$ and $\cup_{i \in I} a_i$ mean the same as the signs $\cap \{a_i | i \in I\}$ and $\cup \{a_i | i \in I\}$, respectively. We shall *not* write $\prod A_i$, $\cap a_i$ or $\cup a_i$ instead of $\prod_{i \in I} A_i$, $\cap_{i \in I} a_i$ or $\cup_{i \in I} a_i$, respectively. If M is a set of sets, $\prod M$ is the set of all mappings t of M such that $t(X) \in X$ for every element X of M. This implies that $\prod M = \prod(X | X \in M) = \prod_{X \in M} X$.

LEMMA 1. *Let g be an element of $\prod(R_f | f \in \prod_{i \in I} A_i)$. Then there exists an element i of I such that $A_i \subseteq R_g$.*

Proof. Assume the contrary. Then $A_i - R_g$ is non-empty for every element i of I. Hence there exists an element f_0 of $\prod_{i \in I} A_i$ such that

$$f_0(i) \in A_i - R_g$$

for every element i of I.

Received October 19, 1964.

This implies that

$$f_0(i) \neq g(f)$$
$$\text{for } i \in I,\ f \in \prod_{j \in I} A_j;$$

hence

$$g(f_0) \neq f_0(i) \text{ for } i \in I,$$

and

$$g(f_0) \notin R_{f_0},$$

contrary to $g \in \prod(R_f; f \in \prod_{\in I} A_i)$.

This proves the lemma.

Henceforth, it is understood that L is a complete lattice.

LEMMA 2.

$$\bigcup_{i \in I} \cap A_i \leqq \cap \left\{ \bigcup_{i \in I} f(i);\ f \in \prod_{j \in I} A_j \right\}.$$

(It should be observed $\cap A_i$ is the meet of the set A_i for a *fixed* i),

Proof. If $i \in I$ and $f \in \prod_{j \in I} A_j$, then $f(i) \in A_i$, and

$$\cap A_i \leqq f(i) \leqq \bigcup_{k \in I} f(k).$$

DEFINITION 1. We say that meets are completely distributive with respect to joins in L if

$$(3) \qquad \bigcap_{X \in M} \cup X = \bigcup_{t \in \Pi M} \bigcap_{X \in M} t(X)$$

or every set M of subsets of L. We say that joins are completely distributive with respect to meets in L if

$$(4) \qquad \bigcup_{X \in M} \cap X = \bigcap_{t \in \Pi M} \bigcup_{X \in M} t(X)$$

for every set M of subsets of L.

If this definition is accepted the following statement becomes a theorem:

If meets are completely distributive with respect to joins in L then

$$(5) \quad \bigcap_{i \in I} \cup A_i = \cup \left\{ \bigcap_{i \in I} f(i) \,\middle|\, f \in \prod_{j \in I} A_j \right\}.$$

If joins are completely distributive with respect to meets in L then

$$\bigcup_{i \in I} \cap A_i = \cap \left\{ \bigcup_{i \in I} f(i) \,\middle|\, f \in \prod_{j \in I} A_j \right\}.$$

The following proof of (2) will include a proof of (5). ((5) does not immediately follow from Definition 1 because the sets $A_i (i \in I)$ need not be all different.)

Let meets be completely distributive with respect to joins in L. Let

$$N = \left\{ R_f \middle| f \in \prod_{i \in I} A_i \right\}.$$

Then (3) holds with M replaced by N. Hence

$$\cap \left\{ \cup R_f \middle| f \in \prod_{i \in I} A_i \right\} = \bigcup_{t \in \Pi N} \bigcap_{X \in N} t(X). \tag{6}$$

Let $t \in \prod N$. Then

$$t(R_f) \in R_f \text{ for } f \in \prod_{i \in I} A_i.$$

Hence

$$\left(t(R_f) \middle| f \in \prod_{i \in I} A_i \right) \in \prod \left(R_f ; f \in \prod_{i \in I} A_i \right).$$

By Lemma 1, there exists an element i of I such that

$$A_i \subseteqq \left\{ t(R\ ) ; f \in \prod_{j \in I} A_j \right\}$$

or, equivalently,

$$A_i \subseteqq \{t(X);\ X \in N\},$$

whence

$$\bigcap_{X \in N} t(X) \leqq \cap A_i.$$

Hence

$$\bigcup_{t \in \Pi N} \bigcap_{X \in N} t(X) \leqq \bigcup_{i \in I} \cap A_i. \tag{7}$$

By (6) and (7),

$$\cap \left\{ \bigcup_{i \in I} f(i) \middle| f \in \prod_{j \in I} A_j \right\} \leqq \bigcup_{i \in I} \cap A$$

Because of Lemma 2,

$$\bigcup_{i \in I} \cap A_i = \cap \left\{ \bigcup_{i \in I} f(i) \middle| f \in \prod_{j \in I} A_j \right\}. \tag{8}$$

Let M be any set of subsets of L. Then, for the case that $I = M$, and $A_X = X$ for $X \in M$, (8) obviously becomes (4). Hence joins are completely distributive with respect to meets in L.

By duality, if joins are completely distributive with respect to meets in L, then the dual of (8) holds, and meets are completely distributive with respect to joins in L.

Hence (2) and (5) hold.

Reference

[1] George N. Raney, *Completely distributive complete lattices*, Proc. Amer. Math. Soc. 3 (1952), 677–680.

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